Factoring Sparse Resultants of Linearly Combined Polynomials

Manfred Minimair*
manfred@minimair.org
http://minimair.org
Department of Mathematics and Computer Science
Seton Hall University
400 South Orange Avenue
South Orange, NJ 07079, USA

ABSTRACT

This paper is part of the author's work on determining the irreducible factors of sparse (toric) resultants of composed polynomials. The motivation behind this work is to use the factors for efficient elimination of variables, by sparse resultant computation, from composed polynomials. Previous works considered the sparse (toric) resultant of polynomials having arbitrary (mixed) supports composed with (i.e. evaluated at) polynomials having the same (unmixed) supports and of polynomials having the same (unmixed) supports composed with polynomials having arbitrary (mixed) supports, resp. Here, we consider the sparse resultant of linear polynomials having arbitrary (mixed) supports composed with polynomials having arbitrary (mixed) supports, also called "linearly combined polynomials", (under a natural assumption on their exponents). The main contribution of this paper is to determine the irreducible factors, together with their exponents, of the sparse resultant of these linearly combined polynomials. This result essentially generalizes a result by Gelfand, Kapranov and Zelevinsky factoring the sparse resultant of unmixed dense linear polynomials composed with polynomials with unmixed supports. It is expected that this result can be applied to eliminate variables from linearly combined polynomials with improved efficiency.

Categories and Subject Descriptors

I.1.1 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Expressions and Their Representation*

*Partially supported by Seton Hall University Research Council and by MITACS in the symbolic analysis project (PI Prof. P. Borwein, Simon Fraser University, Burnaby, Canada).

ISSAC'03, August 3–6, 2003, Philadelphia, Pennsylvania, USA.

General Terms

Theory

Keywords

Composition, Resultant, Linear Combination

1. INTRODUCTION

Resultants are fundamental in solving systems of polynomial equations and therefore have been extensively studied (cf. [11], [24], [6], [4], [7], [14], [20], [25], [8], [21], [29], [23], [10], [2]). Recent research is focused on utilizing structure, naturally occurring in real life problems, of polynomials, for example, sparsity (cf. [31], [13], [12], [9], [5], [32], [3], [28]) as well as composition (cf. [20], [8], [22], [19], [26], [27]).

This paper is part of the author's work on determining the irreducible factors of sparse (toric) resultants of composed polynomials. The motivation behind this work is to use the factors for efficient elimination of variables, by sparse resultant computation, from composed polynomials. Previous works [19] and [27] respectively considered the sparse (toric) resultant of polynomials having arbitrary (mixed) supports composed (i.e. evaluated at) with polynomials having the same (unmixed) supports and of polynomials having the same (unmixed) supports composed with polynomials having arbitrary (mixed) supports. Here, we consider the sparse resultant of linear polynomials having arbitrary (mixed) supports composed with polynomials having arbitrary (mixed) supports, also called "linearly combined polynomials", (under a natural assumption on their exponents). The main contribution of this paper is to determine the irreducible factors, together with their exponents, of the sparse resultant of these linearly combined polynomials.

It is expected that this result can be applied to eliminate variables from linearly combined polynomials with improved efficiency (see Remark 7).

This result essentially (up to a constant factor) generalizes a result by Gelfand, Kapranov and Zelevinsky (Corollary 2.2 on p. 256 of [13]) factoring the sparse resultant of unmixed dense linear polynomials composed with polynomials with unmixed supports. Note that Theorem 1 only generalizes this result up to a constant factor because the constant factor is irrelevant for the computational applications (see Remark 7) motivating the current paper. For the interested reader, we state Gelfand's, Kapranov's and Zelevinsky's result in detail: Let $g_1, \ldots, g_n$ be Laurent polynomials, in the variables $x_1, \ldots, x_{n-1}$ having the same supports $\mathcal{A}$ and let $F = [f_{ij}]$ be an invertible $n \times n$-matrix. Furthermore, let $h_i = \sum_{j=1}^{n} f_{ij} g_j$ and p be the normalized volume of the Newton polytope of the g_j's. Then the unmixed resultant $\mathrm{Res}_{\mathcal{A}}(h_1, \ldots, h_n)$ equals $(\det F)^p \cdot \mathrm{Res}_{\mathcal{A}}(g_1, \ldots, g_n)$.

The reader might wonder whether one can utilize composition structures for other fundamental operations. In fact, this has already been done for some operations. For examples, dense (Macaulay) resultant, Gröbner bases, SAGBI bases, subresultants and Galois groups of certain differential operators have been studied respectively in [26], [17] and [15], [30], [18] and [1] using various mathematical techniques. However, it seems that those techniques cannot be applied to the study of sparse resultants. Therefore in this paper we use mathematical methods that are essentially different from those.

We assume that the reader is familiar with the notions of sparse (toric) resultant, supports of sparse Laurent polynomials, the integer lattice $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ generated by supports $\mathcal{B}_1, \ldots, \mathcal{B}_n$ (i.e. $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ equals the integer lattice generated by $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_{n-1})$ and all differences of elements of $\mathcal{B}_n$), the mixed volume $\mathrm{MV}_L(P_1, \ldots, P_n)$ of polytopes $P_1, \ldots, P_n$ (i.e. the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in the normalized volume of $\lambda_1 P_1 + \cdots + \lambda_n P_n$, normalized with respect to the lattice L) and the normalized volume of Newton polytope (cf. [9], [13], [31]), both normalized with respect to the lattice L (volume normalized with respect to the elementary simplex of the lattice L).

2. MAIN RESULT

Before we state the main theorem we fix a few notations.

We let $f \circ (g_1, \ldots, g_n)$ denote the (Laurent) polynomial obtained from composing the polynomial f with the sparse (Laurent) polynomials $g_1, \ldots, g_n$. For the case of f being sparse linear, which is the subject of this work, we also call $f \circ (g_1, \ldots, g_n)$ a linearly combined (Laurent) polynomial. Furthermore, let $f_1, \ldots, f_n$ be sparse linear (homogeneous) polynomials in the variables $y_1, \ldots, y_n$ with distinct symbolic coefficients with supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and let $g_1, \ldots, g_n$ be sparse Laurent polynomials in the variables $x_1, \ldots, x_{n-1}$ with distinct symbolic coefficients with supports $\mathcal{B}_1, \ldots, \mathcal{B}_n$. (Thus these polynomials are considered as polynomials over the algebraic closure generated by the complex numbers and all these symbolic coefficients.) Moreover, we let $\mathcal{C}_j$ denote the support of $f_j \circ (g_1, \ldots, g_n)$. Furthermore, we let $\mathrm{r}(\omega, \mathcal{B})$ denote the vector (up to positive multiples) $(r_1, \ldots, r_n)$ such that the half space $\left\{e \mid \sum_{i=1}^{n-1} \omega_i e_i \geq r_j\right\}$ supports the convex hull of $\mathcal{B}_j$. (By "half space supporting a polytope" we mean that the whole polytope is contained in the half space and that a face of the polytope is contained in the boundary of the half space.) We also require that ω is a normal vector pointing *inward* this half space. (Note that we also allow $\omega = 0$ considering $\omega = 0$ as the inward normal vector for the whole vector space of all vectors e.) Furthermore, let $f_{\mathcal{A}}^{\mathrm{r}(\omega,\mathcal{B})}$ be the part, whose Newton polytope has normal vector $\mathrm{r}(\omega, \mathcal{B})$, of the polynomial f with support $\mathcal{A}$. Furthermore, let $f|_J$ be obtained from the polynomial f in the variables $y_1, \ldots, y_n$ by replacing, for all j in the set J of indices, the variable y_j in f by 0. Furthermore, we let $\mathcal{A}|_J$ denote the support of $f|_J$, where f has only non-zero coefficients and support $\mathcal{A}$. If $f|_J$ vanishes then $\mathcal{A}|_J$ is the empty set.

Now, we are ready to state the main theorem.

THEOREM 1 (MAIN THEOREM). *Suppose that for all sets J of indices and all vectors ω*

$$\dim \mathcal{L}(\mathcal{A}_1^{\mathrm{r}(\omega,\mathcal{B})}|_J, \ldots, \mathcal{A}_n^{\mathrm{r}(\omega,\mathcal{B})}|_J) \leq |I_{\omega,J}|, \tag{1}$$

where $I_{\omega,J}$ is the set of indices i such that $\mathcal{A}_i^{\mathrm{r}(\omega,\mathcal{B})}|_J \neq \emptyset$.

Then

$$\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}(f_1 \circ (g_1, \ldots, g_n), \ldots, f_n \circ (g_1, \ldots, g_n))$$

equals up to constant factor the product, over vectors ω and sets J of indices, of

$$\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega,\mathcal{B})}|_J, \ldots, \mathcal{A}_n^{\mathrm{r}(\omega,\mathcal{B})}|_J} \left(f_1{}_{\mathcal{A}_1}^{\mathrm{r}(\omega,\mathcal{B})}|_J, \ldots, f_n{}_{\mathcal{A}_n}^{\mathrm{r}(\omega,\mathcal{B})}|_J\right)^{p_{\omega,J}}$$

times

$$\mathrm{Res}_{\mathcal{D}_1,\ldots,\mathcal{D}_n}(g_1, \ldots, g_n)^q.$$

In the above products, if there is more than one irreducible factor for different ω and J, the exponent $p_{\omega,J}$ is, for any $i_0 \in I_{\omega,J}$, the mixed volume, normalized with respect to the lattice $\mathcal{L}\left((\mathcal{C}_i^{\omega})_{i \in I_{\omega,J}}, (\mathcal{B}_j^{\omega})_{j \in J}\right)$, of the polytopes

$$\left(\mathrm{NP}\left((f_i \circ (g_1, \ldots, g_n))_{\mathcal{C}_i}^{\omega}\right)\right)_{i \in I_{\omega,J} \setminus \{i_0\}}, (\mathrm{NP}(g_j{}_{\mathcal{B}^{\omega}}^{\omega}))_{j \in J},$$

and otherwise, the exponent $p_{\omega,J}$ is the mixed volume, normalized with respect to the lattice $\mathcal{L}(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ of the polytopes

$$\mathrm{NP}\left((f_1 \circ (g_1, \ldots, g_n))_{\mathcal{C}_1}^{\omega}\right), \ldots, \mathrm{NP}\left((f_n \circ (g_1, \ldots, g_n))_{\mathcal{C}_n}^{\omega}\right).$$

Furthermore, $\mathcal{D}_j$ is the intersection of all supports $\mathcal{C}_i$ such that f_i is not constant in y_j and q is

$$\frac{\sum_{i_0 \in I} \mathrm{MV}_{\mathcal{L}((\mathcal{C}_i)_{i \in I \setminus \{i_0\}})}\left((\mathrm{NP}(f_i \circ (g_1, \ldots, g_n)))_{i \in I \setminus \{i_0\}}\right)}{\sum_{i_0 \in I} \mathrm{MV}_{\mathcal{L}((\mathcal{D}_i)_{i \in I \setminus \{i_0\}})}\left((g_i)_{i \in I \setminus \{i_0\}}\right)}.$$

EXAMPLE 2. Let's consider the following polynomials.

$$\begin{aligned}
f_1 &:= a_{11}\, y_1 + a_{12}\, y_2, \\
f_2 &:= a_{21}\, y_1 + a_{22}\, y_2, \\
f_3 &:= a_{33}\, y_3, \\
g_1 &:= b_{00} + b_{20}\, x_1^2 + b_{02}\, x_2^2 + b_{22}\, x_1^2 x_2^2, \\
g_2 &:= c_{00} + c_{10}\, x_1 + c_{01}\, x_2 + c_{11}\, x_1 x_2, \\
g_3 &:= d_{00} + d_{10}\, x_1 + d_{01}\, x_2 + d_{11}\, x_1 x_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= \{(1,0,0), (0,1,0)\}, \\
\mathcal{A}_2 &= \{(1,0,0), (0,1,0)\}, \\
\mathcal{A}_3 &= \{(0,0,1)\}, \\
\mathcal{B}_1 &= \{(0,0), (2,0), (0,2), (2,2)\}, \\
\mathcal{B}_2 &= \{(0,0), (1,0), (0,1), (1,1)\}, \\
\mathcal{B}_3 &= \{(0,0), (1,0), (0,1), (1,1)\}.
\end{aligned}$$

One can find two pairs (ω, J), for which $p_{\omega,J} > 0$, namely, $p_{(0,0),\{3\}} = 4$ and $p_{(0,0),\{1,2\}} = 8$. Furthermore, note that

$$
\begin{aligned}
{f_1}_{\mathcal{A}_1}^{(0,0)}|_{\{3\}} &= a_{11}\,y_1 \,+\, a_{12}\,y_2,\\
{f_2}_{\mathcal{A}_2}^{(0,0)}|_{\{3\}} &= a_{21}\,y_1 \,+\, a_{22}\,y_2,\\
{f_3}_{\mathcal{A}_3}^{(0,0)}|_{\{3\}} &= 0,\\
{f_1}_{\mathcal{A}_1}^{(0,0)}|_{\{1,2\}} &= 0,\\
{f_2}_{\mathcal{A}_2}^{(0,0)}|_{\{1,2\}} &= 0,\\
{f_3}_{\mathcal{A}_3}^{(0,0)}|_{\{1,2\}} &= a_{33}\,y_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}_1 = \mathcal{D}_1 \;&=\; \{(0,0),(1,0),(0,1),(1,1),(2,0),(0,2),(2,2)\}\,,\\
\mathcal{C}_2 = \mathcal{D}_2 \;&=\; \{(0,0),(1,0),(0,1),(1,1),(2,0),(0,2),(2,2)\}\,,\\
\mathcal{C}_3 = \mathcal{D}_3 \;&=\; \{(0,0),(1,0),(0,1),(1,1)\}\,.
\end{aligned}
$$

and that $q = 1$. Thus, by Theorem 1, the resultant

$$\operatorname{Res}_{\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3}\left(f_1 \circ (g_1,g_2,g_3)\,, f_2 \circ (g_1,g_2,g_3)\,, f_3 \circ (g_1,g_2,g_3)\right)$$

is up to constant factor the product of

$$\operatorname{Res}_{\mathcal{A}_1,\mathcal{A}_2}\left(a_{11}\,y_1 \,+\, a_{12}\,y_2, a_{21}\,y_1 \,+\, a_{22}\,y_2\right)^4 \;=\; \left(a_{11}\,a_{22} \,-\, a_{12}\,a_{21}\right)^4,$$

$$\operatorname{Res}_{\mathcal{A}_3}\left(a_{33}\,y_3\right)^8 \;=\; a_{33}^8$$

and

$$\operatorname{Res}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3}\left(g_1,g_2,g_3\right).$$

(We do not include the polynomial being the resultant of the g_j's because it is very large.)

REMARK 3. Condition 1 in Theorem 1 guarantees that the resultant of linearly combined polynomials can be factored into factors such that each factor either only contains coefficients of the f_i's or only contains coefficients of the g_j's. The intuitive motivation for this condition is as follows: As is well known, there are certain composed polynomials whose resultant cannot be nicely factored. As an example, consider composed polynomials $f_1 \circ (g_1,g_2,g_3)$ and $f_2 \circ (g_1,g_2,g_3)$ obtained from linear polynomials f_1, f_2 and f_3 in three variables and linear univariate polynomials g_1, g_2 and g_3. Such cases may arise as subsets and in faces (at toric infinity), i.e. among weighted leading forms $f_i \circ (g_1,\ldots,g_n)^{\omega}$, of the composed polynomials considered in the present paper. As an example, consider $f_1 \circ (g_1,g_2,g_3)$, $f_2 \circ (g_1,g_2,g_3)$, $f_3 \circ g_4$, $f_4 \circ g_4$, where g_1, g_2 and g_3 are polynomials in the variable x_1 and g_4 is a polynomial in the variables x_2 and x_3. Condition 1 excludes such composed polynomials.

REMARK 4. It is important to point out that the resultants involving parts of the f_i's in Theorem 1 in fact are certain minors of the coefficient matrix of the f_i's. While it is easy to determine these minors, it is quite lengthy to state them explicitly. Therefore, in order not to make the statement of the theorem more complex, we use the more compact resultant notation.

REMARK 5. Note that $\mathcal{D}_j$ is the intersection of $\mathcal{C}_i$'s which contain $\mathcal{B}_j$. Therefore $\mathcal{B}_j \subseteq \mathcal{D}_j$. Thus the resultant

$$\operatorname{Res}_{\mathcal{D}_1,\ldots,\mathcal{D}_n}\left(g_1,\ldots,g_n\right)$$

of Theorem 1 is not necessarily irreducible. Theorem 1 of [27] can be used to determine its irreducible factors.

REMARK 6. The reader may wonder why the larger supports $\mathcal{D}_j \supseteq \mathcal{B}_j$ (see also Remark 5) arise. It turns out (see the proof of Theorem 1) that $\mathcal{D}_j$ is the largest support such that $f_j \circ (\tilde{g}_1,\ldots,\tilde{g}_n)$, where the $\tilde{g}_i$'s respectively have supports $\mathcal{D}_i$, has support $\mathcal{C}_j$.

REMARK 7. It is expected that this result can be applied to eliminate variables from linearly combined polynomials with improved efficiency. Rather than computing the resultant of the linearly combined polynomials one can compute the irreducible factors individually in a divide-and-conquer-style algorithm and even prune redundant factors (branches in the computation) if desired in specific applications.

3. PROOF OF THE MAIN THEOREM

Before we show the main theorem we prove some auxiliary lemmas. This section can be considered as a generalization of the proof of the main theorem of [19]. We will point out the generalizations and differences in the proofs of the lemmas and Theorem 1 when appropriate.

The first lemma studies the weighted leading form of linearly combined polynomials. It corresponds to Lemma 12 of [19] which studies the weighted leading forms of dense polynomials composed with unmixed polynomials. The case of linearly combined polynomials requires a more detailed consideration of the faces of the outer polynomials f_i.

LEMMA 8. *Let f be a sparse linear (homogeneous) polynomial in the variables $y_1,\ldots,y_n$ with distinct symbolic coefficients with support $\mathcal{A}$. Then*

$$\left(f \circ (g_1,\ldots,g_n)\right)_{\mathcal{C}}^{\omega} \;=\; f_{\mathcal{A}}^{(r_1,\ldots,r_n)} \circ \left({g_1}_{\mathcal{B}_1}^{\omega},\ldots,{g_n}_{\mathcal{B}_n}^{\omega}\right),$$

where $\mathcal{C}$ is the support naturally induced by composition and r_j is such that the half space $\left\{e \,\middle|\, \sum_{i=1}^{n-1} \omega_i\,e_i \,\geq\, r_j\right\}$ supports the Newton polytope of g_j.

PROOF. Note that

$$f \circ (g_1,\ldots,g_n) \;=\; \sum_{\alpha\in\mathcal{A}} a_\alpha g_1^{\alpha_1}\cdots g_n^{\alpha_n},$$

for some symbolic coefficients a_α, where the indices α are tuples of the form $(0,\ldots,0,1,0,\ldots,0)$. Therefore we have

$$\left(f \circ (g_1,\ldots,g_n)\right)_{\mathcal{C}}^{\omega} \;=\; \sum_{\alpha\in\mathcal{A}} a_\alpha \left(g_1^{\alpha_1}\cdots g_n^{\alpha_n}\right)_{\mathcal{C}}^{\omega}.$$

We will see that some of the $(g_1^{\alpha_1}\cdots g_n^{\alpha_n})_{\mathcal{C}}^{\omega}$'s vanish and some equal $(g_1^{\alpha_1}\cdots g_n^{\alpha_n})_{\mathcal{S}_\alpha}^{\omega}$, where $\mathcal{S}_\alpha$ is the support of g_i, where $\alpha_i = 1$.

Since the half space $\left\{e \,\middle|\, \sum_{i=1}^{n-1} \omega_i\,e_i \,\geq\, r_j\right\}$ supports the Newton polytope of g_j, the half space

$$\left\{e \,\middle|\, \sum_{i=1}^{n-1} \omega_i\,e_i \,\geq\, \sum_{j=1}^{n} r_j\,\alpha_j\right\}$$

supports the Newton polytope of $g_1^{\alpha_1}\cdots g_n^{\alpha_n}$. Now, let

$$\left\{e \,\middle|\, \sum_{i=1}^{n-1} \omega_i\,e_i \,\geq\, t\right\}$$

be the half space with inward normal vector ω supporting the Newton polytope of $f \circ (g_1,\ldots,g_n)$. Note that, since the

Newton polytope of $g_1^{\alpha_1}\cdots g_n^{\alpha_n}$ is contained in the Newton polytope of $f\circ(g_1,\ldots,g_n)$, we have $t\le\sum_{j=1}^n r_j\,\alpha_j$, for any $\alpha\in\mathcal{A}$. Therefore $t=\min_{\alpha\in\mathcal{A}}\left(\sum_{j=1}^n r_j\,\alpha_j\right)$ and

$$(g_1^{\alpha_1}\cdots g_n^{\alpha_n})^{\omega}_{\mathcal{C}} \;=\; \begin{cases} 0 & \text{if } t<\sum_{j=1}^n r_j\,\alpha_j,\\ (g_1^{\alpha_1}\cdots g_n^{\alpha_n})^{\omega}_{\mathcal{S}_\alpha} & \text{otherwise.}\end{cases}$$

Further, note that the hyperplane $\left\{e \,\middle|\, \sum_{j=1}^n r_j\,e_j \;=\; t\right\}$ contains the Newton polytope of $f_{\mathcal{A}}^{(r_1,\ldots,r_n)}$. Therefore

$$\begin{aligned}(f\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}} &= \sum_{\alpha\in\mathcal{A}\text{ and } t=\sum_{j=1}^n r_j\,\alpha_j} a_\alpha\,(g_1^{\alpha_1}\cdots g_n^{\alpha_n})^{\omega}_{\mathcal{C}}\\ &= f_{\mathcal{A}}^{(r_1,\ldots,r_n)}\circ\left({g_1}^{\omega}_{\mathcal{B}_1},\ldots,{g_n}^{\omega}_{\mathcal{B}_n}\right).\end{aligned}$$

□

We will use Rojas' Vanishing Theorem (see [32] as well as Lemma 10 of [19]) often throughout this proof. (See also Proposition 2.1 of [13] and Theorem 3.4 of [9] for the unmixed case.) For the convenience of the reader we include the version of Rojas' Vanishing Theorem stated in Lemma 10 of [19]:

> Let $h_1,\ldots,h_n$ be homogeneous Laurent polynomials in the variables $x_1,\ldots,x_n$ with supports $\mathcal{E}_1,\ldots,\mathcal{E}_n$ and with distinct symbolic coefficients. We make the assumption ([9]) that the dimension of the Newton polytopes of the h_i's is $n-1$. For all specializations, with complex numbers, of all the coefficients of the h_j's, we have $\mathrm{Res}_{\mathcal{E}_1,\ldots,\mathcal{E}_n}(h_1,\ldots,h_n)\;=\;0$ iff there is a vector ω such that the leading forms ${h_1}^{\omega}_{\mathcal{E}_1},\ldots,{h_n}^{\omega}_{\mathcal{E}_n}$ of $h_1,\ldots,h_n$ have a common zero in $(\mathbb{C}\setminus\{0\})^n$.

Furthermore, for the convenience of the reader, we summarize a few facts about Rojas' Vanishing Theorem in the following remark. These facts are well known or easy to see and therefore throughout this proof we will utilize these facts without explicitly referring to this remark.

Remark 9. In short, Rojas' Vanishing Theorem ([32], Lemma 10 of [19]) states that the sparse (toric) resultant of Laurent polynomials $h_1,\ldots,h_n$ vanishes iff the h_i's have a common toric root (possibly at toric infinity). By toric root at toric infinity one usually means that certain weighted leading forms of the h_i's (cf. [32], Lemma 10 of [19]) have a common toric root. Equivalently, one also means that the h_i's, after extending their domain to the toric variety naturally constructed from the supports of the h_i's, have a common root in certain non-affine parts of this toric variety (cf. [32]).

In Lemma 10 in [19] it was assumed, that the dimensions of the convex hulls of the supports $\mathcal{E}_i$ of the h_i's is $n-1$. It is easy to see that this restriction on Lemma 10 of [19] can be relaxed. That is, it is sufficient to assume that $\{1,\ldots,n\}$ is the unique subset of $\{1,\ldots,n\}$ essential for $(\mathcal{E}_1,\ldots,\mathcal{E}_n)$, as stated in [32], which also is the necessary and sufficient condition for the existence of a non-trivial sparse resultant. Thus Rojas' Vanishing Theorem as stated in Lemma 10 of [19] can be applied to the composed polynomials in this proof.

Assumption 10. Until the proof of Theorem 1, we assume that for all j,

$$\mathcal{B}_j \;=\; \bigcap_{\substack{i\\ f_i \text{ is not constant in } y_i}} \mathcal{C}_i.$$

As Lemma 13 of [19] the following lemma studies the common roots of certain composed polynomials. Due to the mixed supports of the f_i's there are many more possibilities for how common roots of the composed polynomials can arise. Assumption 10 is critical for controlling the complexity of certain required case distinctions in the following lemma (see its proof).

Lemma 11. *For all specializations, with complex numbers, of all the coefficients of the f_i's and of the g_j's,*

$$\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}(f_1\circ(g_1,\ldots,g_n),\ldots,f_n\circ(g_1,\ldots,g_n))=0$$

implies that

$$\mathrm{Res}_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(g_1,\ldots,g_n)=0$$

or there is a vector ω and a set J of indices such that

$$\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega,\mathcal{B})}|_J,\ldots,\mathcal{A}_n^{\mathrm{r}(\omega,\mathcal{B})}|_J}\left({f_1}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}|_J,\ldots,{f_n}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_n}|_J\right)=0.$$

Proof. Fix f_i's and g_j's such that

$$\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}(f_1\circ(g_1,\ldots,g_n),\ldots,f_n\circ(g_1,\ldots,g_n))=0.$$

Then, by Rojas' Vanishing Theorem (cf. Remark 9) there is a vector ω and a toric $x\in(\mathbb{C}\setminus\{0\})^n$ such that

$$(f_1\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_1}(x)=0,\ldots,(f_n\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_n}(x)=0.$$

Let's fix such a vector ω and a toric x. Then there is a set J of indices such that

$$\begin{gathered}{g_1}^{\omega}_{\mathcal{B}_1}(x)=y_1,\ldots,{g_n}^{\omega}_{\mathcal{B}_n}(x)=y_n,\\ {f_1}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}(y)=0,\ldots,{f_n}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_n}(y)=0,\end{gathered}$$

where $y_j=0$ iff $j\in J$. Therefore there are sets J and $I_{\omega,J}$ of indices such that

$$\begin{gathered}{g_1}^{\omega}_{\mathcal{B}_1}(x)=y_1,\ldots,{g_n}^{\omega}_{\mathcal{B}_n}(x)=y_n,\\ {f_i}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}|_J(y)=0,\ \text{for } i\in I_{\omega,J},\end{gathered}$$

where $y_j=0$ iff $j\in J$ and ${f_i}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}|_J\neq 0$ iff $i\in I_{\omega,J}$.

Now, let U, $V_{\omega,J}$ and $W_{\omega,J}$ be sets of pairs (f,g) such that respectively

- $\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}(f_1\circ(g_1,\ldots,g_n),\ldots,f_n\circ(g_1,\ldots,g_n))=0$,
- $I_{\omega,J}=\emptyset$ and there exists a toric x such that ${g_j}^{\omega}_{\mathcal{B}_j}(x)=0$, for all $j\in J$, and
- $I_{\omega,J}\neq\emptyset$ and there exists a toric y such that ${f_i}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_i}|_J(y)=0$, for all $i\in I_{\omega,J}$,

where $f=(f_1,\ldots,f_n)$ and $g=(g_1,\ldots,g_n)$ are tuples of Laurent polynomials defined over the complex numbers with supports $\mathcal{A}_i$ and respectively $\mathcal{B}_j$. So far we have seen that $U\;\subseteq\;\bigcup_{(\omega,J)\in S}V_{\omega,J}\cup\bigcup_{(\omega,J)\in T}W_{\omega,J}$, for suitable sets S and T.

Next observe that Assumption 10 and Condition 1 respectively imply that the algebraic closures $\overline{V}_{\omega,J}$ and $\overline{W}_{\omega,J}$ of the sets $V_{\omega,J}$ and $W_{\omega,J}$ are of codimension 1 or greater in the space of tuples (f, g) of Laurent polynomials with complex coefficients over the supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$ (cf. also [31], [34]). Furthermore, observe that, since U is a hypersurface, all its irreducible components are of codimension 1 (cf. [33]). Therefore we have

$$U \subseteq \bigcup_{(\omega,J)\in S'} \overline{V}_{\omega,J} \cup \bigcup_{(\omega,J)\in T'} \overline{W}_{\omega,J},$$

where $\overline{V}_{\omega,J}$ is of codimension 1, for $(\omega, J) \in S'$, for respective suitable $S' \subseteq S$. and $\overline{W}_{\omega,J}$ is of codimension 1, for $(\omega, J) \in T'$, for suitable $T' \subseteq T$. By the definition of resultant, $\overline{V}_{\omega,J}$ and $\overline{W}_{\omega,J}$ are sets of pairs (f, g) such that

$$\mathrm{Res}_{(\mathcal{B}_j^\omega)_{j\in J}} \left((g_j{}^{\omega}_{\mathcal{B}_j})_{j\in J} \right) = 0, \text{ and respectively}$$

$$\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega,\mathcal{B})}|_J,\ldots,\mathcal{A}_n^{\mathrm{r}(\omega,\mathcal{B})}|_J} \left(f_1{}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}|_J, \ldots, f_n{}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_n}|_J \right) = 0,$$

for $(\omega, J) \in S'$ and respectively $(\omega, J) \in T'$. Next, by Assumption 10 we see that $S' = \{((0, \ldots, 0), \{1, \ldots, n\})\}$. We have thus proved the lemma. □

LEMMA 12. *Any irreducible polynomial divides the resultant* $\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n} (f_1 \circ (g_1, \ldots, g_n), \ldots, f_n \circ (g_1, \ldots, g_n))$, *being a polynomial in the coefficients of the* f_i*'s and* g_j*'s, iff it divides, for any vector* ω *and set* J *of indices,*

$$\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega,\mathcal{B})}|_J,\ldots,\mathcal{A}_n^{\mathrm{r}(\omega,\mathcal{B})}|_J} \left(f_1{}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}|_J, \ldots, f_n{}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_n}|_J \right)$$

or it divides

$$\mathrm{Res}_{\mathcal{D}_1,\ldots,\mathcal{D}_n} (g_1, \ldots, g_n),$$

where $\mathcal{D}_j$ *is the intersection of all supports* $\mathcal{C}_i$ *such that* f_i *is not constant in* y_j*.*

PROOF. The proof of this lemma utilizes Lemma 11 and Hilbert's Nullstellensatz. It is almost identical to the proof of Lemma 14 of [19]. □

Subsequently, we will determine the exponents on the irreducible factors in the resultant of the linearly combined polynomials. For the case of multiple factors containing coefficients of the f_i's, we take an approach different from [19]. It seems that one cannot easily generalize, to multiple factors, the technique used in [19].

REMARK 13. Subsequently, we will frequently refer to Bernshtein's theorem. By Bernshtein's theorem we mean its toric version of [32], which essentially states that if the Laurent polynomials $h_1, \ldots, h_n$ in n variables with supports $\mathcal{C}_1, \ldots, \mathcal{C}_n$ have finitely many toric roots (possibly at toric infinity) then they have exactly

$$\mathrm{MV}_{\mathcal{L}(\mathcal{C}_1,\ldots,\mathcal{C}_n)} (\mathrm{NP}(h_1), \ldots, \mathrm{NP}(h_n))$$

common toric roots (possibly at toric infinity).

When we write "common root", we mean root in the above sense (compare to Remark 9).

In the following lemma $h_1, \ldots, h_n$ and $p_1, \ldots, p_n$ are Laurent polynomials, in the variables $x_1, \ldots, x_{n-1}$, over supports $\mathcal{E}_1, \ldots, \mathcal{E}_n$, with distinct symbolic coefficients and with coefficients in any field of characteristic root.

LEMMA 14. *The least total degree, i.e. the minimum total degree of any monomial, of* $\mathrm{Res}_{\mathcal{E}_1,\ldots,\mathcal{E}_n} (h_1 + p_1, \ldots, h_n + p_n)$ *in the symbolic coefficients of the* h_i*'s is the number (counting multiplicities) of common toric roots (possibly at toric infinity) of the* p_i*'s.*

PROOF. We assume that the reader is familiar with the notions of multiplicity and blow-up (cf. e.g. [16], [33]).

Note that the least total degree of

$$\mathrm{Res}_{\mathcal{E}_1,\ldots,\mathcal{E}_n} (h_1 + p_1, \ldots, h_n + p_n)$$

in the symbolic coefficients of the h_i's is the multiplicity of 0 on the hypersurface H defined by the equation

$$\mathrm{Res}_{\mathcal{E}_1,\ldots,\mathcal{E}_n} (h_1 + p_1, \ldots, h_n + p_n) = 0,$$

where we consider the symbolic coefficients of the h_i's as the variables of the equation. Therefore we show that the multiplicity of 0 on the hypersurface H is the number (counting multiplicities) of common toric roots (possibly at toric infinity) of the p_i's. We count the multiplicity of 0 by blowing up 0 and by counting the number (including their multiplicities) of the distinct branches, we obtain in the blown up algebraic set. By Rojas' Vanishing Theorem the hypersurface H is the projection π, discarding $x_1, \ldots, x_{n-1}$, of the set Z of tuples $(h_1, \ldots, h_n, x_1, \ldots, x_{n-1})$, such that the $(h_i + p_i)$'s have a common toric root $x_1, \ldots, x_{n-1}$ (possibly at toric infinity). The projection π can be factored into a composed sequence of projections $\pi = \pi_{n-1} \circ \cdots \circ \pi_2 \circ \pi_1$, where $\pi_i : Z_i \to Z_{i+1}$, $Z_1 := Z$ and Z_i, for $i > 0$, is the set of tuples $(h_1, \ldots, h_n, x_i, \ldots, x_{n-1})$, obtained from Z_{i-1} via the natural projection discarding x_{i-1}. Each p_i can be viewed as a blow-up, as described e.g. in [16]. Thus the projection π resolves the possibly singular point 0 of H into distinct points $(0, x_1, \ldots, x_{n-1})$ in Z. By definition of π_i, the number of these distinct points (counting multiplicities) is the number (counting multiplicities) of common toric roots (*possibly at toric infinity*) of the $(0 + \pi_i)$'s, that is, of the π_i's. We have thus proved the lemma. □

Now we determine the numbers $p_{\omega,J}$ occurring in Theorem 1.

LEMMA 15. *If there is more than one irreducible factor for different* ω *and* J*, the exponent* $p_{\omega,J}$ *is, for any* $i_0 \in I_{\omega,J}$*, the mixed volume, normalized with respect to the lattice* $\mathcal{L}\left((\mathcal{C}_i^\omega)_{i\in I_{\omega,J}}, (\mathcal{B}_j^\omega)_{j\in J} \right)$, *of the polytopes*

$$\left(\mathrm{NP}\left((f_i \circ (g_1, \ldots, g_n))^{\omega}_{\mathcal{C}_i} \right) \right)_{i\in I_{\omega,J}\setminus\{i_0\}}, (\mathrm{NP}(g_j{}^{\omega}_{\mathcal{B}^\omega}))_{j\in J},$$

and otherwise, the exponent $p_{\omega,J}$ *is the mixed volume, normalized with respect to the lattice* $\mathcal{L}(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ *of the polytopes*

$$\mathrm{NP}\left((f_1 \circ (g_1, \ldots, g_n))^{\omega}_{\mathcal{C}_1} \right), \ldots, \mathrm{NP}\left((f_n \circ (g_1, \ldots, g_n))^{\omega}_{\mathcal{C}_n} \right).$$

PROOF. We first consider the case when the resultant of the linearly combined polynomials has only one irreducible factor containing the coefficients of the f_i's. For this case, the exponent $p_{\omega,J}$ can be computed proceeding as in the proof of Lemma 16 of [19].

Next we consider the case when the resultant of the linearly combined polynomials has more than one irreducible factor containing the coefficients of the f_i's. We first show that $p_{\omega,J}$ is greater than or equal to

the mixed volume, normalized with respect to the lattice $\mathcal{L}\left((\mathcal{C}_i^\omega)_{i\in I_{\omega,J}}, (\mathcal{B}_j^\omega)_{j\in J} \right)$, of the polytopes

$$\left(\mathrm{NP}\left((f_i \circ (g_1, \ldots, g_n))^{\omega}_{\mathcal{C}_i} \right) \right)_{i\in I_{\omega,J}\setminus\{i_0\}}, (\mathrm{NP}(g_j{}^{\omega}_{\mathcal{B}^\omega}))_{j\in J},$$

Fix suitable ω and J. We choose Laurent polynomials f_i''s with respective supports $\mathcal{A}_i$ such that the ${f_i'}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_i}|_J$'s have exactly one common toric root, this is the only common root of the f_i''s, no $|I|-1$ Laurent polynomials among ${f_i'}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_i}|_J$, for $i \in I$, have a common toric root at toric infinity and

$$\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega',\mathcal{B})}|_{J'},\ldots,\mathcal{A}_n^{\mathrm{r}(\omega',\mathcal{B})}|_{J'}}\left({f_1'}^{\mathrm{r}(\omega',\mathcal{B})}_{\mathcal{A}_1}|_{J'},\ldots,{f_n'}^{\mathrm{r}(\omega',\mathcal{B})}_{\mathcal{A}_n}|_{J'}\right)$$

does not vanish, for any $(\omega', J') \neq (\omega, J)$. Such f_i''s exist because the

$$\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega',\mathcal{B})}|_{J'},\ldots,\mathcal{A}_n^{\mathrm{r}(\omega',\mathcal{B})}|_{J'}}\left({f_1}^{\mathrm{r}(\omega',\mathcal{B})}_{\mathcal{A}_1}|_{J'},\ldots,{f_n}^{\mathrm{r}(\omega',\mathcal{B})}_{\mathcal{A}_n}|_{J'}\right)\text{'s}$$

are irreducible polynomials in the coefficients of the f_i's and because of Rojas' Vanishing Theorem. Note that Lemma 12 says that the resultant of the f_i's composed with the g_j's equals, up to constant factor, the power product

$$\prod_{\omega,J}\mathrm{Res}_{\mathcal{A}_1^{\mathrm{r}(\omega,\mathcal{B})}|_{J},\ldots,\mathcal{A}_n^{\mathrm{r}(\omega,\mathcal{B})}|_{J}}\left({f_1}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_1}|_{J},\ldots,{f_n}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_n}|_{J}\right)^{p_{\omega,J}} \times\ \mathrm{Res}_{\mathcal{D}_1,\ldots,\mathcal{D}_n}(g_1,\ldots,g_n)^q\,.$$

When we replace the f_i's in this power product by $f_i + f_i'$, we see by Lemma 14, that the least total degree, in the coefficients of the f_i's, of the such modified power product is $p_{\omega,J}$. Furthermore, the least total degree of the resultant of the f_i's composed with the g_j's is greater than or equal to the number (counting multiplicities) of common toric roots (possibly at toric infinity) of the $f_i' \circ (g_1,\ldots,g_n)$'s because the resultant, with respect to supports $\mathcal{C}_1,\ldots,\mathcal{C}_n$, of

$$(f_1+f_1')\circ(g_1,\ldots,g_n),\ldots,(f_n+f_n')\circ(g_1,\ldots,g_n)$$

can be obtained from the resultant, with respect to supports $\mathcal{C}_1,\ldots,\mathcal{C}_n$, of

$$h_1+f_1'\circ(g_1,\ldots,g_n),\ldots,h_n+f_n'\circ(g_1,\ldots,g_n),$$

where the h_i's are Laurent polynomials with distinct symbolic coefficients with supports $\mathcal{C}_i$, by replacing h_i by $f_i \circ (g_1,\ldots,g_n)$. Therefore we subsequently show that the number (counting multiplicities) of common toric roots (possibly at toric infinity) of the $f_i' \circ (g_1,\ldots,g_n)$'s is the normalized mixed volume, with respect to the integer lattice $\mathcal{L}\left(\left(\mathcal{C}_i^\omega\right)_{i\in I_{\omega,J}},\left(\mathcal{B}_j^\omega\right)_{j\in J}\right)$, of the polytopes

$$\left(\mathrm{NP}\left((f_i\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_i}\right)\right)_{i\in I_{\omega,J}\setminus\{i_0\}},\left(\mathrm{NP}\left({g_j}^{\omega}_{\mathcal{B}^\omega}\right)\right)_{j\in J}.$$

By definition of the f_i''s and by Lemma 8, the number of common roots of the $f_i' \circ (g_1,\ldots,g_n)$'s equals the number of common roots of

$$\left((f_i'\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_i}\right)_{i\in I},\left({g_j}^{\omega}_{\mathcal{B}^\omega}\right)_{j\in J}.$$

Let's determine the number of common roots of these Laurent polynomials: Fix $i_0 \in I$. Then, by Bernshtein's theorem, and because the ${f_i'}^{\mathrm{r}(\omega,\mathcal{B})}_{\mathcal{A}_i}|_J$'s have only one common root, indeed, the number of common roots of the Laurent polynomials $\left((f_i'\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_i}\right)_{i\in I\setminus\{i_0\}},\left({g_j}^{\omega}_{\mathcal{B}^\omega}\right)_{j\in J}$ is the normalized mixed volume, with respect to the integer lattice $\mathcal{L}\left(\left(\mathcal{C}_i^\omega\right)_{i\in I_{\omega,J}},\left(\mathcal{B}_j^\omega\right)_{j\in J}\right)$, of the polytopes

$$\left(\mathrm{NP}\left((f_i\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_i}\right)\right)_{i\in I_{\omega,J}\setminus\{i_0\}},\left(\mathrm{NP}\left({g_j}^{\omega}_{\mathcal{B}^\omega}\right)\right)_{j\in J}.$$

Next we show that the $p_{\omega,J}$'s are equal to their minima from above, which we denote by $m_{\omega,J}$. It is sufficient to show that the total degree in the coefficients of the resultant of the f_i's composed with the g_j's equals the total degree of the above power product, for $p_{\omega,J} = m_{\omega,J}$. It is sufficient because (obviously) the total degrees of the factors in the above power product are not negative or root and thus the total degree, viewed as a function in $p_{\omega,J}$, is increasing in each $p_{\omega,J}$. If a certain $p_{\omega,J}$ were greater than its minimum then a different $p_{\omega,J}$ would be less than its minimum and this is a contradiction. So let's compare the total degree of the resultant of the linearly combined polynomials to the total degree of the above power product when $p_{\omega,J} = m_{\omega,J}$: By Lemma 1.2 of [34], the total degree of the resultant of the linearly combined polynomials, in the coefficients of the f_i's, is the sum, over $k_0 \in K = \{1,\ldots,n\}$, of the number of common roots of Laurent polynomials $(h_k)_{k\in K\setminus\{k_0\}}$, where h_k has support $\mathcal{C}_k$. Now, let's consider the total degree of the above power product. It is

$$\sum_{J,\omega} m_{J,\omega}\sum_{i_0\in I}1 =$$

$$\sum_{J,\omega}\sum_{i_0\in I}\mathrm{MV}_{\mathcal{L}\left(\left(\mathcal{C}_i^\omega\right)_{i\in I},\left(\mathcal{B}_j^\omega\right)_{j\in J}\right)}\Big(\left(\mathrm{NP}\left((f_i\circ(g_1,\ldots,g_n))^{\omega}_{\mathcal{C}_i}\right)\right)_{i\in I\setminus\{i_0\}},\left(\mathrm{NP}\left({g_j}^{\omega}_{\mathcal{B}^\omega}\right)\right)_{j\in J}\Big).$$

Now, observe that by Bernshtein's theorem and by the construction of J and ω (cf. the proof of Lemma 11), the latter sum equals the sum, over $k_0 \in K = \{1,\ldots,n\}$, of the number of common roots of Laurent polynomials $(h_k)_{k\in K\setminus\{k_0\}}$, where h_k has support $\mathcal{C}_k$. □

The next lemma gives an explicit formula for the quantity q from Theorem 1. This lemma generalizes Lemma 17 of [19].

Note that the current section can be viewed as a generalization of the section containing the proof of the main theorem of [19] (see also the remarks at the beginning of the current section). For the convenience of the reader familiar with [19] we structured the proof presented in the current section such that the reader can easily see the differences and similarities. Therefore we state the next lemma, generalizing Lemma 7 of [19], even though we leave out its proof because it is essentially identical to the proof of Lemma 7 of [19].

LEMMA 16. *The exponent q is*

$$\frac{\sum_{i_0\in I}\mathrm{MV}_{\mathcal{L}\left((\mathcal{C}_i)_{i\in I\setminus\{i_0\}}\right)}\left((\mathrm{NP}(f_i\circ(g_1,\ldots,g_n)))_{i\in I\setminus\{i_0\}}\right)}{\sum_{i_0\in I}\mathrm{MV}_{\mathcal{L}\left((\mathcal{D}_i)_{i\in I\setminus\{i_0\}}\right)}\left((g_i)_{i\in I\setminus\{i_0\}}\right)}.$$

PROOF. The proof is essentially identical to the proof of Lemma 17 of [19]. □

Now we are ready to prove the main theorem.

Proof of theorem 1 (Main theorem):

Now, we relax Assumption 10, which we had assumed for the previous auxiliary lemmas.

Let $\overline{g}_j := g_j + \tilde{g}_j$, where $\tilde{g}_j$ is a Laurent polynomial with distinct symbolic coefficients distinct from all the other symbolic coefficients in this paper, such that the support of $\overline{g}_j$ is $\mathcal{D}_j$. Then Theorem 1 of [27] implies that

$$\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}(f_1\circ(g_1,\ldots,g_n),\ldots,f_n\circ(g_1,\ldots,g_n))$$

equals

$$\mathrm{Res}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}\left(f_1\circ(\overline{g}_1,\ldots,\overline{g}_n),\ldots,f_n\circ(\overline{g}_1,\ldots,\overline{g}_n)\right)$$

after substituting $\tilde{g}_1 = 0, \ldots, \tilde{g}_n = 0$.

Thus Lemma 12, Lemma 16 and Lemma 15 imply the main theorem. □

4. CONCLUSION

This paper considered the sparse resultant of linear polynomials having arbitrary (mixed) supports composed with polynomials having arbitrary (mixed) supports, also called linearly combined polynomials, under a natural assumption on their exponents. The main contribution of this paper is to determine the irreducible factors, together with their exponents, of the sparse resultant of these composed polynomials. This result essentially generalizes a result by Gelfand, Kapranov and Zelevinsky factoring the sparse resultant of unmixed dense linear polynomials composed with polynomials with unmixed supports.

Future directions of research include considering *non-linear* polynomials with arbitrary supports composed with polynomials with arbitrary supports.

Acknowledgment

I thank Prof. Cox, Amherst College, as well as Prof. Sottile, University of Massachusetts, for inviting me to Amherst, MA, USA, where part of this work was conducted. I also thank Prof. P. Borwein and Prof. Monagan for inviting me, during the summer of 2002, to Simon Fraser University, Burnaby, Canada, where part of this work was conducted. I also thank my department at Seton Hall University, South Orange, NJ, USA, and the Seton Hall University Research Council for partially supporting this work.

5. REFERENCES

[1] P. H. Berman and M. F. Singer. Calculating the Galois group of $L_1(L_2(y)) = 0$, L_1, L_2 completely reducible operators. *J. Pure Appl. Algebra*, 139(1-3):3–23, 1999. Effective methods in algebraic geometry (Saint-Malo, 1998).

[2] L. Busé, M. Elkadi, and B. Mourrain. Generalized resultants over unirational algebraic varieties. *J. Symbolic Computation*, 29(4-5):515–526, 2000.

[3] J. Canny and I. Emiris. A subdivision-based algorithm for the sparse resultant. *J. ACM*, 47(3):417–451, May 2000.

[4] J. Canny, E. Kaltofen, and Lakshman Y. Solving systems of non-linear polynomial equations faster. In *Proc. ACM-SIGSAM 1989 Internat. Symp. Symbolic Algebraic Comput. ISSAC'89*, pages 121–128. ACM, 1989.

[5] E. Cattani, A. Dickenstein, and B. Sturmfels. Residues and resultants. *J. Math. Sci. Univ. Tokyo*, 5(1):119–148, 1998.

[6] A. Cayley. On the theory of elimination. *Cambridge and Dublin Math. J.*, 3:116–120, 1848.

[7] M. Chardin. *Contributions à l'algèbre commutative effective et à la théorie de l'élimination*. PhD thesis, Université Paris VI, 1990.

[8] C. C. Cheng, J. H. McKay, and S. S. Wang. A chain rule for multivariable resultants. *Proceedings of the American Mathematical Society*, 123(4):1037–1047, April 1995.

[9] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*. Springer Verlag, New York, Berlin, Heidelberg, 1998.

[10] C. D'Andrea and A. Dickenstein. Explicit formulas for the multivariate resultant. *Pure Appl. Algebra*, 164(1-2):59–86, 2001.

[11] A.-L. Dixon. The eliminant of three quantics in two independent variables. *Proc. London Math. Soc.*, 7:49–69, 473–492, November 1908.

[12] I. Emiris and V. Pan. The structure of sparse resultant matrices. In *Proc. Int. Symp. on Symbolic and Algebraic Computation (ISSAC)*. ACM Press, 1997.

[13] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston, 1994.

[14] L. González-Vega. Une théorie des sous-résultants pour les polynômes en plusieurs variables. *C. R. Acad. Sci. Paris Sér. I Math.*, 313/13:905–908, 1991.

[15] J. Gutierrez and R. Rubio San Miguel. Reduced Gröbner bases under composition. *J. Symbolic Computation*, 26(4):433–444, 1998.

[16] Joe Harris. *Algebraic Geometry*. Springer-Verlag, New York, Berlin, Heidelberg, 1992.

[17] H. Hong. Subresultants under composition. *J. Symbolic Computation*, 23(4):355–365, 1997.

[18] H. Hong. Groebner basis under composition I. *J. Symbolic Computation*, 25(5):643–663, 1998.

[19] H. Hong and M. Minimair. Sparse resultant of composed polynomials I. *J. Symbolic Computation*, 33:447–465, 2002.

[20] J. P. Jouanolou. Le formalisme du résultant. *Adv. Math.*, 90(2):117–263, 1991.

[21] D. Kapur and T. Saxena. Sparsity considerations in Dixon resultants. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, pages 184–191, New York, 1996. ACM.

[22] D. Kapur and T. Saxena. Extraneous factors in the Dixon resultant formulation. In *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI)*, pages 141–148, New York, 1997. ACM.

[23] R. Lewis and P. Stiller. Solving the recognition problem for six lines using the Dixon resultant. *Math. Comput. Simulation*, 49(3):205–219, 1999.

[24] F. S. Macaulay. *The algebraic theory of modular systems*. Cambridge Mathematical Library, 1916.

[25] D. Manocha and J. Canny. Multipolynomial resultant algorithms. *J. Symbolic Computation*, 15(2):99–122, 1993.

[26] M. Minimair. Dense resultant of composed polynomials. Accepted by J. Symbolic Computation, 2002.

[27] M. Minimair. Sparse resultant of composed polynomials II. *J. Symbolic Computation*, 33:467–478, 2002.

[28] M. Minimair. Sparse resultant under vanishing coefficients. Accepted by J. Algebraic Combinatorics, 2002.

[29] G. Nakos and R. M. Williams. Elimination with the Dixon resultant. *Mathematica for Education and Research*, 6/3:11–21, 1997.

[30] P. Nordbeck. SAGBI bases under composition. *J. Symbolic Computation*, 33(1):67–76, 2002.

[31] P. Pedersen and B. Sturmfels. Product formulas for resultants and Chow forms. *Mathematische Zeitschrift*, 214:377–396, 1993.

[32] J. M. Rojas. Solving degenerate sparse polynomial systems faster. *J. Symbolic Computation*, 28(1 and 2):155–186, July/August 1999. Special Issue Polynomial Elimination – Algorithms and Applications.

[33] I. R. Shafarevich. *Basic Algebraic Geometry*, volume 1. Springer-Verlag, Berlin Heidelberg New York, 2nd edition, 1994.

[34] Bernd Sturmfels. On the Newton polytope of the resultant. *Journal of Algebraic Combinatorics*, 3:207–236, 1994.